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Technologies to Enhance and Extend Children's Understanding of Geometry: A Configurative Thematic Synthesis of the Literature

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ABSTRACT

Empirical evidence indicates that students are not learning geometry with relational understanding of the concepts. Studies have shown that digital technologies can support students in mathematics. The purpose of this study was to find which technologies and technological affordances are specific to learners of geometry. This paper presents the results of a configurative thematic synthesis of empirical studies and theoretical papers to show that dynamic geometry environments (DGEs: including 3D DGEs) and logo-based environments were the main types of technologies used to support geometry learners. The results of this study also reveal that there are five main technological supports provided to geometry learners – visualization, manipulation, cognitive tools, discourse promoters, and ways of thinking.

Keywords

Mathematics, Dynamic Geometry Environments, Visualization, Manipulation, Cognitive tools, Discussion promoters

Introduction

Geometry is the study of properties, relationships, and transformations of spatial objects, within an interconnected network of concepts and representational systems. Spatial reasoning undergirds geometry, enabling students to cognitively construct and manipulate mental representations of those spatial objects (Clements & Battista, 1994). Therefore, concomitant study of geometry and spatial reasoning should take place (Battista, 2007). The study of geometry provides students with opportunities to better understand the physical environments in which they live. Unfortunately, mathematics is often taught through lecture, which affords surface understanding, merely requiring students to memorize mathematical facts. The epistemological nature of geometry is called into question, as the findings of copious studies (e.g., Ubuz & Üstün, 2004) indicate that students are not learning geometry with relational understanding of the concepts (Skemp, 1976).

For students to develop a full understanding of the geometry, they must take an active role in the learning process (Piaget & Inhelder, 1967; Vygotsky, 1978). The learner-centered philosophy, stemmed from Bruner's (1966) discovery learning and since then has included constructivist learning, constructionist learning, and socio-constructivist learning. In a dual progression, throughout the learner-centered pedagogical epoch, technology was becoming affordable and accessible for use in schools, and educators witnessed the explosive growth of technologies available for the teaching and learning of geometry (Fey et al., 1984). These technologies provided a way for students to take that active role in learning geometry.

The purpose of this study is to find out what types of digital technologies can be used to support the learning of geometry and what affordances the digital technologies can provide to geometry students. To that end, the two questions guiding this study are:

- What types of digital technologies support the learning of geometry?
- How can the use of digital technologies enhance and extend children's understanding of geometry?

Literature review

Configurative thematic synthesis

A systematic review is the art and science of identifying, selecting, and synthesizing studies to provide a comprehensive and trustworthy picture of the topic being studied (Oakley, 2012). There are two broad modes of synthesis: aggregation and configuration (Sandelowski, Voils, Leeman, & Crandlee, 2011). Aggregation syntheses are conducted by counting the numbers of studies and particular components of those studies. Configuration studies as those that generate new theories with the studies similar to pieces in a mosaic slotted together to form a gestalt image (Sandelowski et al., 2011). In this study, a configuration approach is used to determine what technologies are available for learning about geometry and how they support the learner.

A thematic synthesis is intertwined with a configurative approach. It is a systematic approach to bringing a variety of *findings* together to provide a perspective on a particular theme that emerges. Those findings can be empirical studies and theoretical papers (Thomas & Harden, 2008). This study employs the configurative, thematic synthesis approach to answer the two research questions guiding this study.

Past reviews

Laborde, Kynigos, Hollebrands, and Strässer (2006) conducted a review of teaching and learning geometry with technology over three decades. This synthesis of research was organized studies using four categories; (a) the nature of geometry mediated by technology, (b) technology and the learning of geometry, (c) the design of tasks, and (d) the use of geometry by teachers. Zbiek, Heid, Blume, and Dick (2007), published a study about technology in mathematics education. This study was not focused on geometry in particular, but specifically referenced technological tools and technology-based mathematical activities, students' behavior in the context of technology, teaching issues related to technology in mathematics education, and effects of technology on mathematics curriculum content. The most recent synthesis of the literature was conducted by Sinclair and Robutti (2013) and was focused on two areas: the evolution of proof in school mathematics and the impact on the kinds of research questions and studies over the last decade; and the epistemological and cognitive nature of dragging and measuring, and their connection to proof.

These prior syntheses reported about positive influence that technology has on teaching and learning of geometry. However, the purpose of this study is to identify exactly what those positive benefits are. In this synthesis the researchers initially provide an overview of the technologies that are available for learning geometry concepts and then a configurative thematic synthesis methodology is used to identify specifically what those technologies do that extend and enhance students' understanding in this area.

Methodology

To answer the two research questions guiding this study, a configurative thematic synthesis (Thomas, Harden, & Newman, 2012) of the literature was conducted. Empirical studies and theoretical papers primarily from the 21st Century were selected from the intersection of geometry and technology. Literature was identified using electronic and *hand searching* (Brunton, Stansfield, & Thomas, 2012). The hand searching process included recommendations from scholars and historically literatures recognized for yielding foundational findings or theory with sustained relevance. The literature for this review is illustrative, not exhaustive.

The literature was coded using an inductive process. Several themes emerged from deduction. The use of inductive and deductive reasoning is common practice in a thematic synthesis (Thomas et al., 2012). Line by line coding revealed core themes, followed by axial coding that yielded: (a) two types of technologies – Dynamic Geometry Environments (DGEs) and logo-based environments; and (b) five ways that these technologies extend and enhance geometric learning – visualization, manipulation, cognitive tools, discourse promoters, and ways of thinking. An example of part of the intricate coding process can be found in this extensive mind map (see http://bit.ly/1nF2CE3).

Six subject experts (three technology and three mathematics professors from two research intensive universities in the south west of the United States) selected the 105 papers based on their fundamental theoretical and empirical value. Of the 105 papers, 45 were omitted due to redundancy. In other words, if multiple scholars agreed on a particular point, only one or two of the most recognized scholar names were included in this manuscript. 60 papers were included in this configurative thematic synthesis.

Findings

Types of technologies

As many different types of geometry software flooded the market, a trend toward two types of computer environments have developed: Dynamic Geometry Environments (DGEs, which includes 3D Dynamic Geometry Software); and Logo-based environments. There are approximately 70 different two-dimensional and three-dimensional DGEs available worldwide; the majority of these programs are based on the Geometer's Sketchpad, Cabri-Géomètre, Geometric Supposer, and Thales (Laborde et al., 2006).

Two-dimensional DGEs appeared in the 1980's and provide students with objects (e.g., lines, points, circles) and basic tools (e.g., for drawing perpendicular lines from a specific point) to create composite figures. Also, dynamic transformations can be performed, and the ability to trace actions for later visual inspection. Three-dimensional DGEs were introduced in the mid-2000 with software such as Cabri3D and Geogebra3D. This software allows learners to observe the movement of objects from different perspectives, while maintaining its properties.

Logo is a basic programming language used for programs such as Logo-based Turtle Geometry (TG) and the related Microworlds. TG typically involves a robotic turtle directed around the screen using commands; as the turtle moves, it draws lines creating various shapes. Microworlds are computational environments in which students can engage in exploration and construction activities (Sarama & Clements, 2002). More recent versions of Logo-type geometry programs use virtual reality learning environments (VRLEs), which provide students opportunities to study three-dimensional shapes via a variety of semiotic resources (Yeh & Nason, 2004).

In the last decade, mobile technologies, such as smart phones and tablet computers, offering new types of DGEs, TG, and Microworlds for students on mobile devices. Using geometry programs on mobile devices provide additional opportunities for mathematics to be learner-centered. Students can learn geometry on mobile devices pertaining to particular concepts using real-life artifacts to solve authentic problems (Traxler, 2011). In addition, unlike mouse operated computers, students interact with mobile devices using multi-touch screens, allowing them to directly manipulate geometric objects.

Enhancing and extending students' learning

To consider the role of technology in mathematics requires an understanding of two different types of activities, technical and conceptual (Zbiek et al., 2007). The technical dimension describes mechanical or procedural performance; the ways students interact with technologies to construct, manipulate, and measure geometric figures. While performing technical tasks, they are developing sequences of mathematical actions. Conceptual activities involve students understanding, communicating, and developing mathematical connections, relationships and structures (Zbiek et al., 2007). Although a dichotomy between the two activities has been described, students need to engage in both activities for technology to positively influence student learning (Borwein, 2005).

With consideration for technical and conceptual activities, this configurative thematic synthesis reviewed empirical and theoretical findings to collate a record of the different ways DGE and Logo environments promote students' geometric understanding. The findings were organized into five broad categories – visualization, manipulation, cognitive tools, discourse promoters, and ways of thinking.

Visualization

Visual representation

Computer graphics provide visual representations that can promote learning beyond those representations within traditional instruction (Clements & Battista, 1994). This claim is supported by empirical findings; Fesaki, Sofronious and Mavroudi (2011) found that preschool children benefit from computer representations when learning about shapes. Many cultural influences, such as picture books, text books, posters, toys, and school supplies, typically provide students with rigid conceptions of shapes, which are detrimental to hierarchical shape categorizations (Clements, 2004). Researchers reported that students' continual exposure to rigid shape depictions, misconceptualized non-essential attributes of shapes as important (Burger & Shaughnessy, 1986).

Computer programs provide visual representations of shapes with greater variety of graphical and symbolic components, and afford construction and manipulation tools (Ben-Zvi, 2000). Thus enabling students expanded repertoire of representations, beyond prototypical shape depictions (Zbiek et al., 2007).

The *multirepresentational software* (Ben-Zvi, 2000) provided by geometry programs, is described by Kaput (1992) as *action notation systems* that involve construction, using calculations and transformations versus being limited to visual interpretation only. As students have the ability to create and manipulate objects, this assists students in perceiving objects as geometric entities, rather than just visual objects (Zbiek et al., 2007). Therefore, students are more likely to reflect on properties for categorizing shapes, as they are able to simultaneously take into account the specific and grounded with the abstract and generalized. As students come to recognize properties of shapes, this facilitates their movement from the van Hiele level one to level two (van Hiele, 1984); from perceiving shapes as gestalt-like unanalyzed visuals to understanding that properties uniquely define shapes (Battista, 2007).

As students create visual representations using geometry software, the process provided opportunities to externalize their mathematical ideas; they externalized intuitive expectations. Once intuitions are translated as shapes on computer screens, the act is obtrusive and students may reflect on what they produced (Papert, 1980). Students drawing shapes using pencil and paper may not be required to provide descriptions as they are for geometry programs. For instance, on the computer, the square's properties must be explicitly articulated via commands or actions to produce a square. Geometry environments also allow representations to be saved for later analysis and reflection, and they can be edited, manipulated, transformed, separated into parts, or combined (Ben-Zvi, 2000). Miyazaki et al. (2012) posited that three-dimensional DGEs connect better with the real world than geometrical constructions in two dimensional DGEs. They argued that three-dimensional DGE explorations encouraged student dispositions for recognizing examples of geometry within the real world.

Spatial visualization

Spatial visualization is the skill that enables students to understand and perform imagined movements of objects in two-dimensional and three-dimensional spaces (Gutiérrez, 1992). This skill is different than visual thinking, as the image is more abstract, malleable, and less focused than a fully formed picture (Giaquinto, 2007). The students' interaction between the mechanical (spatial), and the theoretical (geometrical) supports the development of spatial reasoning (Laborde et al., 2006). The interconnected nature of spatial-graphical and theoretical reasoning are made explicit through active manipulation of objects and studies have recorded students' spatial/theoretical development while using DGEs (viz., Clements & Sarama, 2007). The three-dimensional DGEs fully encourage development of spatial/theoretical understanding with the ability to create mathematical models of real-world objects and their dynamic movements clarifying that correspondence between the real phenomenon and mathematical structure (Miyazaki et al., 2012).

Manipulation

The manipulation of objects plays a significant role for developing spatial visualization, and it also makes other substantial contributions to geometrical understanding. Students' initial representations are developed through action (Battista, 2007), and students need opportunities to manipulate mathematical objects to develop overarching geometric and spatial understandings (Kamina & Iyer, 2009). Computer programs can provide representations that are as real and personally meaningful to students as physical representations (Sarama & Clements, 2009). Furthermore, the programs offer significantly more flexibility than physical representations (i.e., concrete materials), allowing changes to be made to best meet the educational needs of students. The size and shape of the objects can be changed, altering all, or just some of the components (Sarama & Clements, 2009). For instance, a computer generated geoboard could be adjusted to include additional pegs, or the overall shape of the geoboard can be changed. One of the key elements within DGEs is the drag feature; the mathematical counterpart to drag would be variation (Laborde et al., 2006). Within DGEs, students are able to drag elements of an object and the visual display provides fluid motions to reflect changes, while maintaining the geometric relationships used to construct object. Therefore, when students drag one element of the shape, it is modified proportionately to maintain geometric properties of the construction. The shapes quasi-independence from the student is a feature of DGEs that is likely positively impact student geometrical understanding (Battista, 2009; Yu, Barrett, & Presmeg, 2009).

Battista (2007) posits that students do not initially recognize properties of shapes, even through technology enabled dragging. This development exemplifies student's geometric thinking at level one of the van Hiele (1984) model – the figure is recognized as a gestalt-like entity. As students continue to interact using the drag feature of DGEs, Battista postulated that students start to notice constraints on shapes, then conceptualize constraints as regularities or invariants, and finally constraints are conceptualized as formal geometric properties of shapes. Paper-and-pencil shapes can be altered, although they often become distorted as students try to make the shape correspond with their expectations (Laborde et al., 2006). In a study involving Cabri3D, researchers found that students were able to explore and formulate conjectures and verify them through proof (Mammana, Micale, & Pennisi, 2012). This encouraged an atmosphere of mathematical exploration.

Computer software can allow students to engage in ways that are not easily duplicated using physical manipulatives (Sarama & Clements, 2009). Within the elementary Common Core State Standards (CCSSO/ NGA, 2010) in geometry, students are required to partition shapes into parts with equal areas. Olive and Lobato (2008) make the case that compared with other methods, partitioning can be performed easier and with more precision using computer tools. They argued that to partition shapes, students must mentally dissembled parts from the whole. With physical materials it is not possible to remove a part from the whole without destroying the original whole…the child has to mentally unitize one part of the whole while maintaining the unity of the whole and compare these two abstracted units. With static pictures the part is either embedded in the whole or is drawn separate from the whole…the child has to compare the separate units while imagining that one is embedded in the other. Using a computer tool that provides the child the ability to dynamically pull a part out of a partitioned whole while leaving the whole intact, the child can enact the disembedding operation that is necessary to make the part-to-whole comparison. (Olive & Lobato, 2008, p. 6)

Sarama and Clements (2009) also described the ways shapes can be partitioned into other shapes. For example, a regular hexagon can be cut into two trapezoids. Using computer tools to manipulate shapes brings the geometric motions conducted by students to an explicit level of awareness (Clements & Sarama, 2007), and with this awareness, students can further mathematize their actions. However, students can forget the sequence of their actions. Many geometry programs can record and replay sequences of actions; this feature supports recall and affords opportunity for students to reflect on past actions (Sarama & Clements, 2009).

Real-world contexts enable students to develop a solid base for understanding geometry (Clements & Sarama, 2007). Mobile devices provide the opportunity to develop ideas within a real-world environment rich in architectural and natural geometrical formations. For example, using tablet devices (e.g., iPad), students within DGEs can take photographs of real-world artifacts and use tools to measure angles. Research findings indicate that students are able to understand geometric concepts easier with real-world connections, as the concepts were more interesting, familiar, and logical to the students (Duatepe-Paksu, 2009). Logo-based Microworlds were developed to represent real-world environments; they offer students opportunity to manipulate objects, make them move to specific commands, and construct other representations.

Cognitive tools

Cognitive tools are defined as technologies that act as external aids to amplify students' cognitive capacities during thinking, learning, and problem solving (Lajoie & Azevedo, 2006). Other terms have been used to name these tools; Pea (1987) described them as *cognitive technologies*; Zbiek et al. (2007), as *cognitive technological tools*; and Hoyles (1995), as *computational scaffolding*. Hoyles and Noss (2003) used the term *expressive tools*, to specifically refer to DGE environment tools. Visualization and manipulation play a significant part in enhancing students' cognitive processes. However, this section specifically describes geometry software as a cognitive tool.

The tools provided within DGEs, Logo, and other similar geometry programs, provide students with a way to access the mathematical characteristics underlying geometry and spatial reasoning (Laborde et al., 2006). The software tools become an extension of the students' thinking once students begin to use the programs. Hoyles (1995) described this extension as computational scaffolding; a support process to aid in constructing situated abstractions. "The software tools exploited by the students provide them with the hooks they need on which to hang their developing ideas" (Hoyles, 1995, p. 5). The tools affect the ways students' think about and solve tasks.

As students create or access visual representations within software, cognitive tools act as a *user agent* (Kaput, 1992) to perform geometric actions under student direction (Zbiek et al., 2007). Cognitive fidelity refers to the degree that the computer's supported method reflects the students' independent method for solving a task (Dick, 2007). Researchers have concluded geometry software can provide cognitively faithful manipulation of objects, and is more faithful and easier than methods involving physical tools (see Olive & Lobato, 2008).

Cognitive tools have the potential to enhance and extend students' learning of geometry in several ways. Pea (1987) described one way as amplifying intellectual activity – computers can increase the speed of mathematical tasks with higher accuracy. In addition, students can work with tools within geometry software to support discerning regularities, which might otherwise remain hidden. Meagher (2006) extended Pea's theory to two-way amplification, he reiterated how students can be amplified by technology, but also described the way students amplify the technology as they refine educational goals so the technology provides the best fit for the goals. Vérillon and Rabardel (1995) postulated, there is a difference between the artifact and an instrument; the software or device (artifact) should not be regarded as a tool (instrument) as it only becomes that after someone appropriates it as such. This transformation into a tool is called *instrumental genesis* which indicates the bi-directionality in which this process takes place, which is towards the self and towards outside reality. Vérillon and Rabardel (1995) describe that in instrumental genesis, the artifact has to be integrated into a person's cognitive structure which is a complex process connecting to the artifacts affordances and constraints and the knowledge of the user.

Pea (1987) also elucidated the role computers have as *reorganizers*. As a reorganizer, technology can bring about structural changes to students' cognitive and sociocultural operations (Ben-Zvi, 2000). Technology can provide novel representations, uncover geometric concepts, and offload tedious or time consuming tasks (Sherman, 2002). Kaput (1992) also highlighted the benefit of offloading routine computations, suggesting that it compacted and enriched students' learning experiences. Ben-Zvi (2000)proposed several ways mathematical software can reorganize students' activities, including: (a) tools shifting students' activity to higher levels, as they integrate tasks and focus attention on detailed planning; (b) tools changing objects and form of activities; (c) tools focusing activities on transforming and analyzing representations; (d) tools supporting situated cognitive modes of thinking and problem solving; and (e) tools enabling students' constructing conceptual meanings by using representative ambiguity.

Feedback is another crucial component in advancing students' geometric understanding. For cognitive tools to be effective they must react in response to student actions and provide clear observable consequences to their actions, and the cognitive tools of geometry software provide such feedback to students through clear visuals that are direct and immediate (Clements, Battista, & Sarama, 2001). Physical tools do not react to students' actions via feedback, and often mistakes can go unnoticed or be misinterpreted by students (Zbiek et al., 2007). In addition, computer environments provide a view of students' conceptions and understandings. Researchers reported designing such technological contexts that prevent students from hiding what they do not know, unlike traditional learning activities, which can mask misconceptions and misunderstandings (Clements & Battista, 1994). Therefore, geometric technologies may support greater opportunities for educators to plan appropriate tasks and activities to fill gaps that can be clearly identified and enhancing students' geometric understanding.

Discourse promoters

The feedback computers provide can act as a catalyst for large or small group discussions (Mariotti, 2000). Students need opportunities to interact with others and share mathematical ideas and findings to develop rich understanding (Chaplin, O'Connor, & Canavan-Anderson, 2009). Computers foster mathematical discourse, augmenting communication from teacher-to-student, or computer-to-student, to a richer student-to-student communication (Van de Walle & Lovin, 2006). In addition, interactive geometry software allows discussion of geometric objects in a manner that was once impossible with traditional paper and pencil representations (Yu et al., 2009).

Computers enable students to produce detailed external representations of their internal mental representations. Once externalized, they are visible phenomena that can be shared and discussed with others. Although the representations are idiosyncratic, the visuals and computer activities provide a common context for students to effectively share their ideas (Yu et al., 2009), and the mediating function of the computer could be used to facilitate communication based on shared language and context. Students can have difficulty describing geometric transformations. As students execute processes of creating and/or moving objects using geometric technology, they can become aware of and mathematize their actions. This awareness, afforded by technology, could be used to help students describe procedures or the motions they have enacted.

Discourse enhances students' learning when students collectively reflect on mathematical ideas (Van de Walle & Lovin, 2006). Geometry software provides instant feedback to students. When software images do not perform as expected, for example the turtle moves in a different direction, students typically want to know why; technology use may be an impetus for some questions that may not come up naturally. Working in pairs or small groups, students can reflect and discuss reasons for unexpected phenomena. The features embedded in the software can be used to undo and repeat actions, or study command sequences to check for accuracy. DGEs offer students interesting arenas to push boundaries of geometric understandings, and in doing so can create disequilibrium that need resolution, and providing opportunities for discourse.

Clements and Battista (1994) highly recommend the use of large monitors or projectors to enable students to see technology displays during whole class discussions. This can increase the already high motivation of cooperative, meaningful computer explorations in geometry with students of all ages and abilities. The record and replay feature within geometry software can be used to incorporate student work into group discourses that might include learning from errors, uncovering misconceptions, and justifying solutions.

The necessity of a justification for the solution comes from the need to explain why a certain construction works (that is, passes the dragging test). This need is reinforced through collective discourses, when different solutions are compared, by validating one's own construction in order to explain why it works and/or to foresee whether or not it will function (Mariotti, 2000, p. 32). Computers can be used to call explicit attention to students' actions and understandings encourage students to re-examine and refine their mathematical understandings through discourses with educators and peers.

Ways of thinking

van Hiele (1984) emphasized that successful students construct networks of relationships, linking geometrical concepts and processes, not isolated facts, rules, and names. Corresponding with learner-centered philosophies, thinking is altered by experiences and interaction with mathematical phenomena (Piaget, 1967), thus enabling students to become *mathematical thinkers* (Soto-Johnson, Cribari, & Wheeler, 2009). In addition, Papert (1980) reported that computers provide environments that allow students engage as mathematicians, instead of environments that teach children about mathematics. However, this epistemological shift towards students being mathematical thinkers is not only a shift for educators, but also for students.

Computers can facilitate the development of autonomous learning (rather than seeking authority), and towards positive beliefs about the creation of mathematical ideas (Clements & Battista, 1994). Students are provided with computational scaffolding within geometry software, encouraging students to explore possibilities, rather than declaring themselves "stuck." Borwein and Bailey (2003) listed several activities in which students' thinking processes were positively influenced by using technology, including (a) gaining insight and intuition, (b) discovering new patterns and relationships, (c) testing and falsifying conjectures, (d) exploring reasonableness of results through proof, (e) suggesting approaches for formal proof, and (f) replacing lengthy hand derivations.

Qualitative findings have highlighted the conceptual change in students' thinking after using Logo (viz., Clements et al., 2001). The study showed that without additional intervention, the geometry program tools enabled students to become more cognizant of their thinking in regards to geometrical ideas, and more analytic, precise and general. In the more recent Diedro-3D study, students reported preferring to use the DGE to explore shapes rather than listening to explanations by the teacher. The program allowed students to think mathematically while working within an environment that provided context that the teacher's lecture could not (Martin-Gutiérrez, Gil, Contero, & Saorin, 2013).

Connecting geometry concepts to the real-world setting is another shift in thinking which can be enhanced with the use of technologies. Gainsburg (2008) described an urgency for connections between mathematics and the real-world, and how many students fail to make this connection. Technologies, such as geometry software, can assist students in making this bridge. Computers provide a connection to the real-world using physical representations of objects and environments (Sarama & Clements, 2009). With the use of mobile computing platforms, students can interact with real-life artifacts (Traxler, 2011). Researchers reported that students using mobile devices for learning mathematics, not only noticed mathematics in the real-world, but they were "turning into mathematicians who looked for real-life phenomena to investigate mathematically" (Daher, 2011). This shift in thinking is not the norm experienced by most students in traditional learning spaces.

Conclusion

A good milieu for the emergence of mathematical knowledge is an environment providing the relevant combination of technological tools, problem situations and mathematical discourses (Kordaki & Potari, 2002). Technologies, such as DGEs can provide both the technical and conceptual activities needed for students to fully understand mathematical concepts. From this review of empirical and theoretical evidence, it is clear that technological affordances are far reaching, and provide a way to extend and enhance children's understanding in geometry. The findings of this configurative thematic synthesis show that technology promotes students' knowledge and understanding of geometry in five ways; visualization, manipulation, cognitive tools, discourse promoters, and new ways of thinking.

Within *visualization*, both visual representation and spatial visualization are highlighted as technology provides a way for students to experiment with objects beyond their non-technological counterparts. Students have the opportunities to visualize the shapes as geometric entities (Zbiek et al., 2007). *Manipulation* was a theme that emerged from this study as being particularly enhanced with technology. For example, students can manipulate objects while they keep their geometrical properties. This is often impossible with concrete manipulatives and paper-and-pencil shapes can be altered to what the student Brunton perceive to be correct (Laborde et al., 2006). *Cognitive tools* was described by Lajoie and Azevedo (2006) as ways that technology can act as an external aid to amplify students' cognitive capacities during thinking, learning, and problem solving. Examples from this study show that technologies can allow a student to use the tools to access mathematical characteristics underlying geometry and spatial reasoning.

Mathematical discourse is a valuable practice in enhancing student understanding. From the findings it appears that technology can enhance that understanding with the undo and repeat actions, options to study command sequences, and other similar technological options. The *new ways of thinking* was the final theme articulated in this study. Compared with the other themes, it appears that this is relatively less topic discussed in the literature and appears to be a new finding emerging from this study. New ways of thinking describes how students shift to become mathematical thinkers. This includes how students can engage as mathematicians in the technological environments (Papert, 1980), the thinking processes described by Borwein and Bailey (2003), and by providing students with context that would not be available in a teacher's lecture (Martin-Gutiérrez et al., 2013).

These five categories connect with a learner-centered framework as the student is actively involved in creating cognitive networks of relationships that link geometric concepts and processes (Clements & Battista, 1994). However, despite the potential that technology can have on learning, technology integration in mathematics is proceeding far slower than one may have predicted (Jones, 2011). Hopefully, the evidence provided in this paper can act as a catalyst to establish a dialectical link between the research findings and approaches to the teaching and learning of geometry in schools.

As we move towards learner-centered pedagogies, it would be prudent for researchers to explore opportunities to better contextualize mathematics to enable students to personally connect to the mathematical concepts. Therefore, further studies should involve more recent technologies, such as the virtual reality learning environments that enable students to study three-dimensional shapes via a variety of semiotic resources. Furthermore, learning with mobile devices, such as cell phones and tablets is a relatively new tool. It would be interesting to see how these 21st Century devices can be used to further contextualize learning and allow students to connect with mathematics in the real world learning rather than try to replicate 20th Century pedagogies.

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