

Using Computer-Assisted Multiple Representations in Learning Geometry Proofs

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ABSTRACT

Geometry theorem proving involves skills that are difficult to learn. Instead of working with abstract and complicated representations, students might start with concrete, graphical representations. A proof tree is a graphical representation of a formal proof, with each node representing a proposition or given conditions. A computer-assisted learning environment called MR Geo is proposed to help students in learning to do theorem proving, with the help of multiple representations including problem description, static figure, dynamic geometry figure, formal proof and proof tree. Empirical results indicated that medium-achievement students enjoyed most in interacting with these representations and found them most helpful in learning geometry proofs while low-achievement students changed their attitudes of hating geometry theorem proving.

Keywords

Multiple representations, Web-based interaction system, Proof tree, Geometry education, Dynamic geometry

Background

When a teacher discusses a geometry proof problem in class, it generally involves oral presentation of a formal proof and body movements pointing at different parts of the figure of the problem. Students must watch, listen, jot notes, and think as a lecture proceeds. They have to refer to many elements of the instruction and incorporate them into their memory (Sweller, 1988). This often causes cognitive overload and poses a negative effect on students' learning. Numerous researchers have experimented different ways of teaching and found serious problems in geometry learners: incomplete comprehension of the problem and mathematical symbols, producing proofs based on direct visual elements (e.g., Chazan, 1993; Healy & Hoyles, 2000), lacking strategic knowledge in producing proofs, etc. Addressing the difficulties in learning geometry, Duval (1998) and Healy and Hoyles (1998) explained that geometry instruction is often more complex than that of numerical operations or elementary algebra. Undoubtedly, geometry education poses a challenge to both teachers and students.

One characteristic of human intelligence is the use of different types of representation. NCTM (1989, 2000) has advocated a K-12 curriculum that stresses mathematical connections among multiple representations. Janvier (1987a) used an interesting analogy to explain the application of multiple representations in teaching: when using multiple representations to illustrate a mathematical concept, it is like a star-like iceberg. The sharp angle of each star indicates a representation, and the students' learning objective is to construct an entire star-like iceberg. Teaching, meanwhile, is to let the star-like iceberg emerge from the water so that students can flexibly operate each representation. Similarly, we believe learning of geometry can be made less difficult by employing multiple representations about geometry proofs. Before presenting these specific representations, we first review how other math topics can be learned with multiple representations.

Behr et al. (1983) built on Bruner's (1966) representation theory to propose an interactive model using a representational system to improve the learning of mathematics. Problem solving would be an easier task if a student is aware of how the structures of different representations are interactively connected. The teaching of multiple representations should be emphasized. As for mathematics education, Janvier (1987b) proposed that instruction on functions could make use of verbal description, table, formula, and graph representations. Kieran (1993) proposed similar representations of functions: situation (graphical or textural description), table, diagram, and algebraic equations. QUADRATIC (Wood & Wood, 1999), using the area of squares to make salient the geometric properties of algebraic expressions, was designed to teach student about equivalences implicit in mappings between algebraic and geometric representations. Wong et al. (2007a) studied the use of representations of natural language, telegraphic

description and diagram in problems about area and circumference of geometric shapes for students of elementary school.

In the mid-1980s to the 1990s, some computer programs are used in teaching and exploring geometry in school. In this study, we divided those computer systems into two categories, one focused on Dynamic Geometry Environment (DGE), which used for exploration; the other is Expert System (ES) or Intelligent Tutoring System (ITS), which is used for problem-solving tasks. Some popular DGEs include Geometer's Sketchpad (GSP; Jackiw, 1997), Cabri Geometry II (<http://www.cabri.com/v2/pages/en/index.php>), Geometry Expert (Chou et al., 1996), and Cinderella's Café (<http://www.cinderella.de/tiki-index.php>). While these DGEs share one common focus on dynamic geometry figures, Geometry Expert also does automatic theorem proving. In a dynamic geometry figure, students can drag a geometry object such as a vertex of a triangle and change the figure dynamically while preserving the given conditions of the figure and geometric invariants, which are the consequences of the given conditions. Thus these programs are commonly used to demonstrate geometry theorems.

A well-known ITS Geometry Proof Tutor (GPT) was developed at Carnegie Mellon University to support students in writing Euclidean proofs, and results showed that the tutors were roughly half as effective as a human tutor (Epstein & Hillegeist, 1990). A similar problem-solving environment is ANGLE, where students can construct graphical representations of Euclidean proofs. Unlike GPT, ANGLE is based on a cognitive model that models and therefore facilitates thinking with an expert-like representation of target knowledge (Koedinger & Anderson, 1993b). While GPT and ANGLE help students produce a complete proof, a web-based system is designed to enhance students' reading and learning of proofs in this study.

Multiple Representations

Some scholars (Lesh, Post & Behr, 1987; Ainsworth, 1999; Duval, 1998) noted that students can learn to do translation between multiple representations and make transitions within each representation. As a result, students can modify incorrect concepts and conjectures by comparing multiple perspectives provided by different representations. However, some researchers have found that even though teachers alert students to the importance of understanding each type of representation, students can fail to notice regularities and discrepancies between representations (Kaput, 1987; de Jong et al., 1998; Kozma, 2003). Sweller (1988) pointed out that when learning with several representations, learners are required to relate disparate sources of information, which may cause a heavy cognitive load that leaves few resources for actual learning. Seufert (2003) noted that learners with little prior knowledge are unable to acquire knowledge from multiple representations. She recommended the provision of more semantic aids in order to reduce such cognitive load.

In this study, we adopt the DeFT (**D**esign, **F**unctions, **T**asks) framework of learning with multiple representations (Ainsworth, 1999, 2006; Ainsworth & Van Labeke, 2004; Van Labeke & Ainsworth, 2001). The DeFT framework refers to different pedagogical functions that multiple representations can play, the design parameters unique to learning with multiple representations and the cognitive tasks undertaken by a learner interacting with multiple representations. There are three main functions of multiple representations: complement, constrain and construct. Multiple representations complement each other by supporting complementary processes or conveying complementary information. Two representations constrain each other, as one supports interpretation of the other. Finally, multiple representations can be used to encourage learners to construct a deeper understanding of a situation.

When learning with multiple representations, learners have to deal with complex cognitive tasks Van Labeke & Ainsworth (2001) and van der Meij & de Jong (2006). First of all, learners have to understand the syntax of each representation. Second, learners must also understand which parts of the domain are represented. Third, learners have to relate the representations to each other whether the representations present the same information. Fourth, learners have to translate between the representations. As Ainsworth (2006, p. 195-196) argued, "*DeFT clarifies the pedagogical functions that multiple representations serve, the often-complex learning demands that are associated with their use and in so doing aims to consider the ways that different designs of multi-representational systems impact upon the process of learning.*"

Although there are many empirical studies of computer-based learning environment based on multiple representation theory (e.g. Wood & Wood, 1999; Van Labeke & Ainsworth, 2001; van der Meij & de Jong, 2006; Brenner, et al.,

1997; Hsu, 2006), geometry proof learning is seldom discussed from the perspective of such theory. In this study, we use the DeFT framework to develop a computer-based geometry proof learning environment and investigate its effects on learners.

Representations for Geometry Problems and Proofs

Healy and Hoyles (1998) and Duval (2002) emphasized that in order to construct a geometry proof, students should first identify the given information and then search for critical hints. They should then apply appropriate geometric properties to conduct deductive reasoning, organize all deductive steps, and write down the sequence of steps inferring from the given conditions to the conclusion. Various representations can be used to specify a geometry problem, support the reasoning process, and represent the final proof.

A geometry problem is specified with a verbal description, often accompanied by a figure. As Mayer and Sims (1994) pointed out, students can build more referential connections when verbal and visual materials are presented contiguously than when they are presented separately. Clements and Battista (1992) also noted that for a student to successfully prove a theorem, the student must build semantic links between the concepts of geometry and the features of a figure. Through bi-directional connections, students can clearly demonstrate the interrelation between the geometric components in a verbal description and the objects in an accompanied figure. Schnotz's (2002) integrative model of text (descriptive representation) and picture (depictive representation) comprehension emphasizes that good graphic design is crucial for individuals with low prior knowledge who need pictorial support in constructing mental models.

Duval (1998) proposed that learning geometry involves three kinds of cognitive processes: visualization, construction, and reasoning. The foundation of reasoning lies in mathematical language and logic. Although the visualization process can sometimes provide individuals with heuristics in proving a theorem, a figure can be misleading sometimes because students are often influenced by paradigmatic images. Therefore, simple-minded visualization can interfere with the process of deduction. For example, students may take a rotated square as a "non-square rhombus." Duval argued, however, "*these three kinds of cognitive process are closely connected and their synergy is cognitively necessary for proficiency in geometry*" (p. 38). In other words, reasoning and visualization processes can complement each other. But, in order to avoid the influence of the paradigmatic images, we can make use of a dynamic geometry figure. Duval (2000) suggested that only through the coordination of different representations could students develop geometry problem solving skills. Previous studies indicate that three types of representations should be useful for learning geometry proofs. The following presents a detailed explanation of these representations.

Problem Representation

The first type of representation is the problem itself, or the problem representation (Fig. 1), which is generally expressed as a text. The problem representation specifies some given conditions and a goal that needs to be proved. Students need to understand the mathematical symbols and language, and the logical relationship between the given conditions and the goal condition.

Visual Representation

The second type of representation is visual representation (Fig. 1), which can be a static or dynamic figure. A static figure groups together relevant information from a problem representation. The correspondence between a text and its figure can provide complementary information for students, which could enhance their comprehension of the geometry problem. A proof involves inferences from the given conditions to a conclusion with a sequence of deductions and a simple figure provides little help for students to construct a proof. However, students can visually check whether a proposition is valid in a figure. In other words, static figures can serve the role of constraining an interpretation.

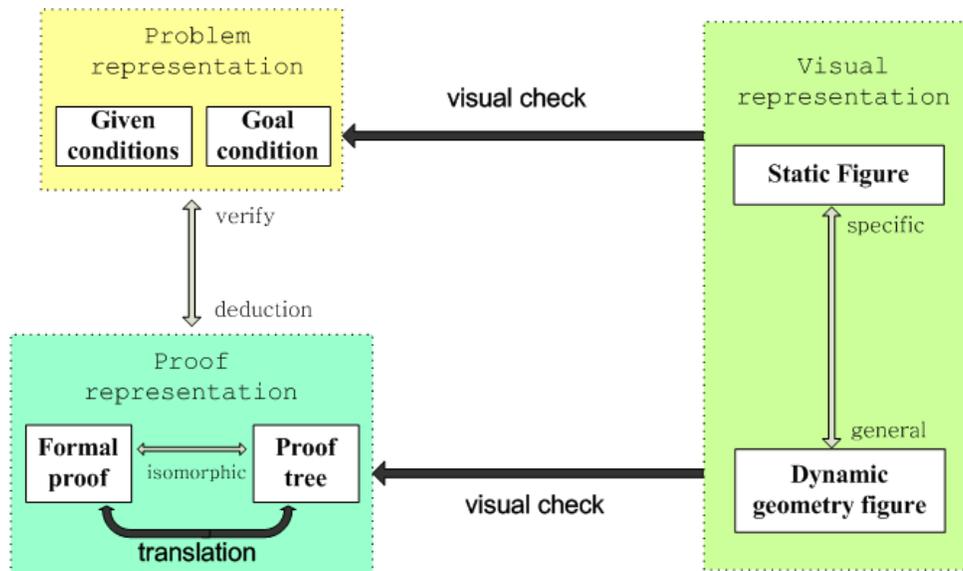


Figure 1. Multiple representations to assist learning of geometry proofs

A dynamic geometry figure is also a visual representation. A student can interact with a dynamic figure, which provides a clear picture of abstract mathematical ideas through concrete object dragging. By manipulating a dynamic figure and observing how it changes, students may be able to avoid over-generalization of theorems from paradigmatic images. DGE allows students to either falsify propositions or enhance the degree to which propositions are believable. The invariant nature of the geometric properties of a dynamic figure can constrain the interpretation of the propositions in a formal proof.

Proof Representation

The third type of problem representation is that of a proof (Fig. 1), which can be a formal proof or a proof tree. As the most rigorous form of solution in geometry proof, a formal proof is composed of a sequence of deductive steps. Formal proof is also the standard deductive reasoning format that instructors and students use to communicate with each other. As Yang and Lin (2008) pointed out, when reading a proof, a learner needs to recognize the role of an example figure, identify the given conditions and the goal condition, generate implicit hypothesis or properties, and apply inference rules. Indeed, the comprehension of a proof is a complicated cognitive process. In order to reduce the cognitive load of proof comprehension on learners, we propose the use of a proof tree.

A proof tree (Fig. 2) is another representation of a proof. ANGLE (Koedinger & Anderson, 1993a) and Matsuda & VanLehn (2005) utilize a tree-like structure to represent a network of inferences. As a visual representation of a formal proof, a proof tree is a hierarchy of nodes inferring from some given conditions to the goal condition (Wong, et al., 2007b). The root node of a proof tree is the goal condition to be proved. The tree consists of leaf nodes and derived nodes. Each node of a proof tree represents a proof step. Each leaf node is a given fact or self-evident condition, e.g., a segment's length equals that of itself. A proof tree shows the logical structure underlying a common textbook proof. In a proof tree, the logical relation between the given facts and the goal is made explicit graphically.

In Fig. 2, the top is the description of a problem, the lower left is a formal proof, and the lower right is a proof tree. In a top-down approach, the proof tree presents the proof process and highlights the logic relationship between the given conditions and the goal. By studying a proof tree, students can check the entire structure of the inferences leading to the goal.

Addressing the difficulties that students encounter in learning geometry, this study proposes a multimedia learning environment to let students interact with multiple representations relevant to a geometry proof. This interactive learning environment is called MR Geo (Multiple Representations for Geometry).

Given: Parallelogram ABCD
with diagonal \overline{AC}
Prove: $\triangle ABC \cong \triangle CDA$

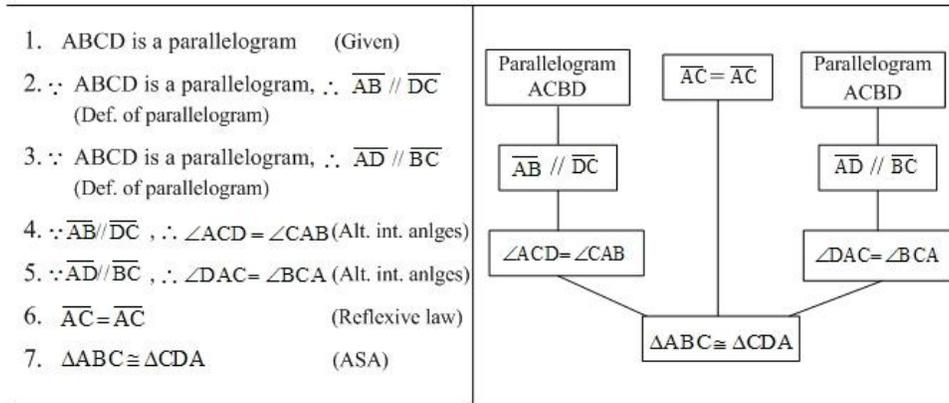
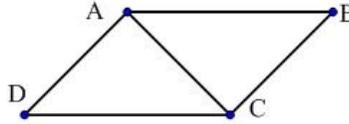


Figure 2. A formal proof and its proof tree

MR Geo

Consider the graphical user interface of MR Geo. There are four frames with five representations relevant to geometry proof (Fig. 3). These representations include problem description, static figure, dynamic geometry figure, formal proof, and proof tree.

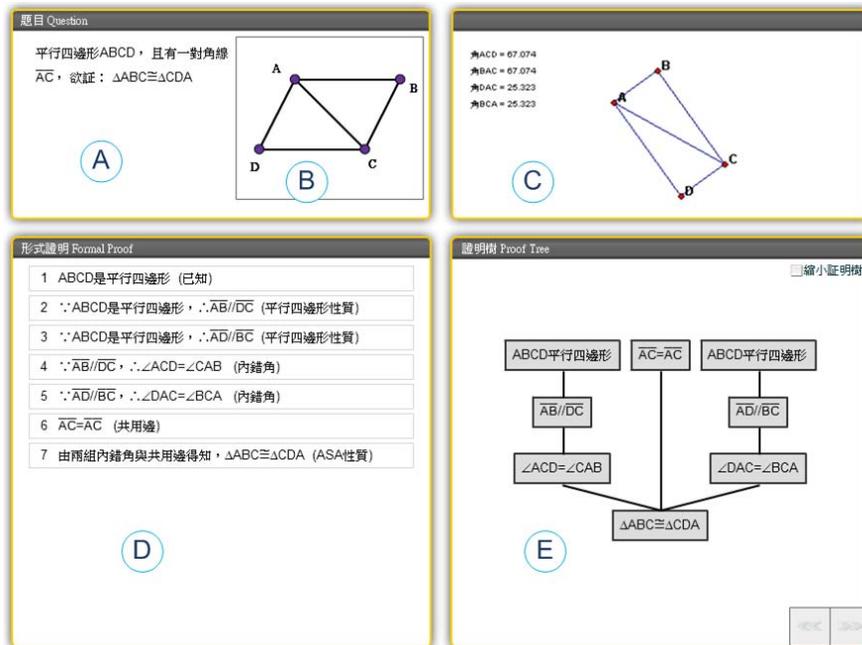


Figure 3. A screenshot of MR Geo

Problem description (A) and static figure (B): In Fig. 3, the top left frame of the interface shows the verbal description a geometry problem accompanied by a static figure. When a user clicks on the problem text, formal proof or proof tree, the clicked objects or the objects in a proof step will be highlighted in the figure with different colors. This design can enhance users' understanding of the correspondence between the figure and other representations.

Dynamic geometry frame (C): The top right frame shows an embedded GSP's Java applet. Students can drag geometric objects in the figure with the mouse. This will cause the values of some chosen attributes of this figure to change dynamically and the values will be displayed at the same time. In this way, students can observe the invariant properties of the dynamic geometry figure and test the validity of each proof step visually.

Formal proof frame (D): The bottom left frame shows a formal proof. A proof is presented in a two-column format. Each row specifies the conditions, which can be given or propositions derived earlier, for deriving a proposition with an inference rule.

Proof tree frame (E): The bottom right frame shows a proof tree. With a click on any node on the proof tree, a student can see the corresponding inference step highlighted in the formal proof. This will help students understand the correspondence between the linear, formal proof and the proof tree. The proof tree clearly shows the relationship between each parent and its children nodes.

In MR Geo, when an object (such as verbal, proposition, and tree's node) is selected in one representation, the corresponding objects in the other representations will be highlighted. For example, consider the problem "Given a parallelogram ABCD with diagonal \overline{AB} . To prove: $\triangle ABC \cong \triangle ADC$ ". When students chooses the problem's given condition "Parallelogram ABCD" in the problem representation, the line segments \overline{AB} , \overline{BC} , \overline{CD} and \overline{AD} will be highlighted with the same color in the static figure. In the formal proof, the first row of "ABCD is a parallelogram" would also be marked, as well as the two "Parallelogram ABCD" nodes in the proof tree. MR Geo provides simultaneous flashing and color-coding to support effective dissemination of information. If students doubt the validity of $\angle ACD = \angle BAC$, she can drag any vertex of the parallelogram on the dynamic representation to check whether the condition is always true.

Purpose of the Study

Given the success of using multiple representations for math instruction in past studies, this study investigates how students would react to the use of multiple representations such as proof tree, formal proof and dynamic geometry figure when they read a proof and construct a proof in the system. In particular, there are several questions we would like to answer. Which representations students prefer when they perform tasks related to theorem proving in MR Geo? After using MR Geo, will students' attitudes towards learning geometry undergo any change? Which types of students benefit more from multiple representations in MR Geo?

Methodology

In order to understand the effect of MR Geo on junior high school students' learning of geometry proofs, we conducted an experiment that spanned six weeks. This experiment used materials on parallelograms and triangles from textbooks and reference books for grade nine mathematics that covered basic algebraic concepts (such as equality axiom), properties and elements of geometry (such as perpendicular bisectors, angle bisectors), and similarity and congruence of triangles.

Participants

The participants were 96 ninth grade students selected from three classes in two public junior high schools in southern Taiwan. There were forty-seven males and forty-nine females with ages ranging from 14.5 to 16.0 (average 15.01). They had already learned basic geometric concepts and elementary proofs. All participants possessed the basic skills to work with a computer. Moreover, all participants did not use any computer-assisted learning system in the learning of geometry and its proof.

As a pretest, we adopted Standard Progressive Matrices Plus (or SPM+; Raven, 2004) to assess students' figure deduction skill. The SPM+, which is a standardized test for students in Taiwan, keeps the 60-item, format of the

Classic SPM. Its test-retest reliability was .87 and split-half reliability was .80-.86. Also, its correlation with math grades was .65. Overall speaking, the reliability and validity of SPM+ are satisfactory.

Based on their mean scores of SPM+ and math grades with equal weighting, the students were divided into three groups: high-achievement group (or HG; 28 persons), medium-achievement group (or MG; 40 persons), and low-achievement group (or LG; 28 persons). By splitting the students into three groups, we can find out which types of learning problems would be encountered by each group of students.

Materials

Research tools used in this study included MR Geo and a questionnaire survey. We designed four tasks of practice and testing for learners in MR Geo, progressing from simple tasks to complex tasks. In simple tasks, most information was provided for the learners while in complex tasks, little information was provided and more efforts were expected from the learners. The four tasks are shown in Fig. 4.

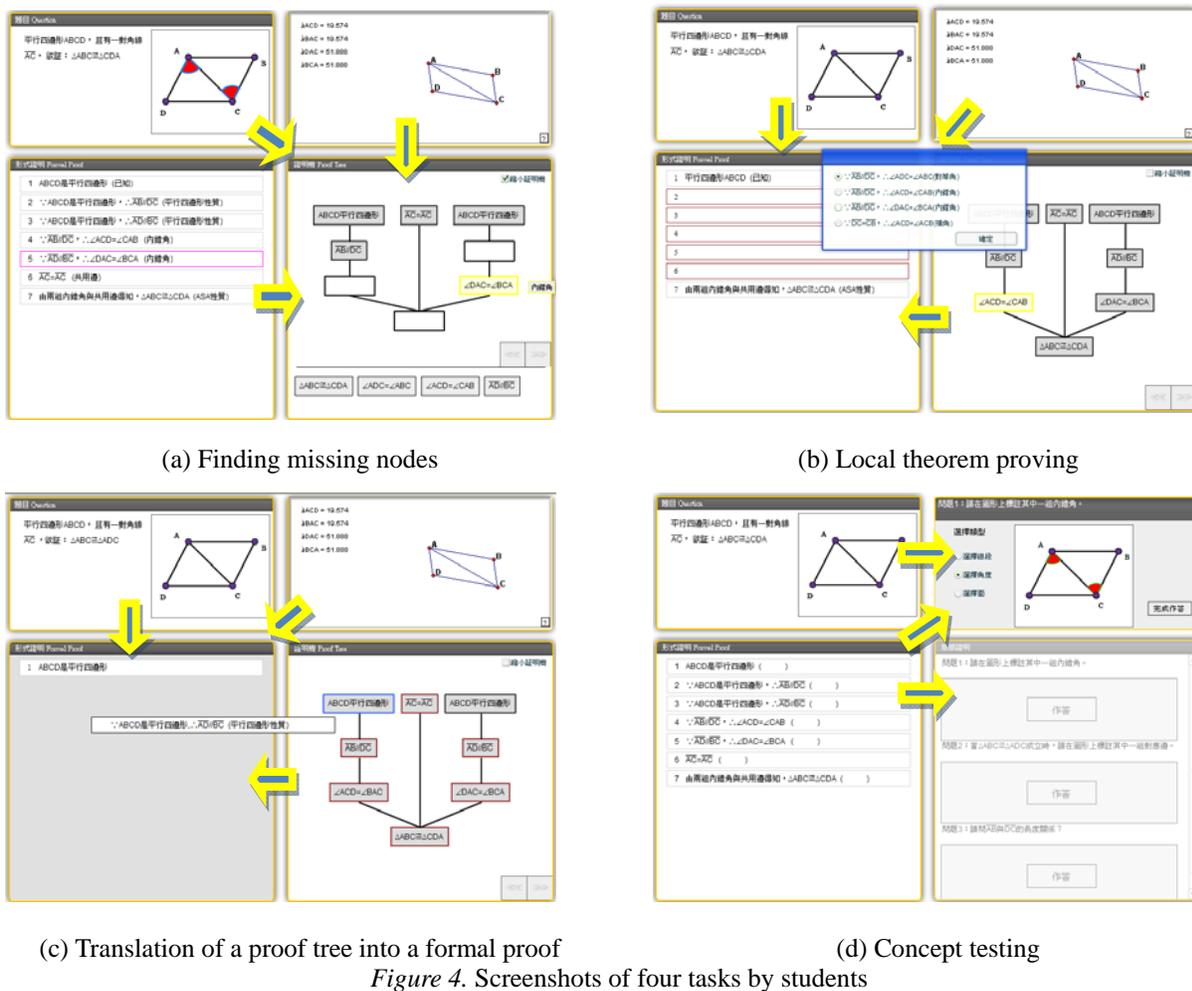


Figure 4. Screenshots of four tasks by students

Task 1 – Finding missing inference steps: To recover the missing nodes of a proof tree, the students can interact with the problem, the formal proof, the static figure, or the dynamic geometry figure (see Fig. 4(a)). By observing the mutual correspondence among these representations, the students might improve their comprehension of the formal proof. In particular, if a student can translate the formal proof into a proof tree, she should be able to recover the missing nodes.

Task 2 – Local theorem proving: Some steps in the formal proof are missing. In the form of multiple-choice questions (select one of 4 choices), the system asks students to fill in the missing deductive steps. In Fig. 4(b), students can obtain relevant information by reading the proof tree and dragging the dynamic figure to check candidate propositions. The main purpose of this task is to help students complete a formal proof after they have read and understood the proof tree.

Task 3 – Translation of a proof tree into a formal proof: Students are asked to construct a formal proof by dragging nodes of the proof tree and drop into rows of the formal proof frame (see Fig. 4(c)). Moreover, they can reorder the rows by dragging them around. In this way, they can explore different orderings of the steps. They can interact with the problem representation or the visual representations whenever they want to check any geometric features or propositions. The major purpose of this task is to improve students’ understanding of how to write a complete formal proof.

Task 4 – Concept testing: The main purpose of this task is to test students’ basic geometric concepts. Students are asked to pick geometric objects from a figure or answer multiple-choice questions. Consider the question in Fig. 4(d) as an example: “Please indicate on the figure a pair of alternate interior angles in the top right frame of the screen.” Students can use the mouse to select a geometric object such as “line”, “angle” by clicking at the object on the static figure. In Fig. 4(d), $\angle DAC$ and $\angle BCA$ are marked by a student as a pair of alternate interior angles.

Results and Discussion

Students’ Interactions with Representations in MR Geo

When students took a test with MR Geo in the last week of the experiment, they could interact with any of the five representations by pointing the mouse at a geometric object or a proof step. Table 1 shows how many times students interacted with the representations. An interaction was counted if the mouse stayed inside the frame of representation for over 2.5 seconds. The data showed that students interacted with proof tree most frequently (2.73 times), followed by formal proof (2.61 times).

Table 1. Students’ average frequency of interacting with representations

Representation	Overall
Problem	1.51
Dynamic geometry figure	.77
Formal proof	2.61
Proof tree	2.73

Also, students interacted less frequently with problem representation (1.51 times) and dynamic geometry figure (.77 times). In problem representation, students usually spent little time to review the question after they have read and understood the given conditions and goal condition. Although students could consult the static figure to check given conditions, they were unable to manipulate the static figure freely like dynamic geometry figure. The interaction frequency of dynamic geometry figure was significantly lower than the other three representations.

Table 2. ANOVA test of students’ frequency of interaction with representations

Group	N	Sum	Average	Variance	F	Scheffé test
HG	28	3089	110.32	149.86	7.79*	MG>HG
MG	40	4795	119.88	134.68		MG>LG
LG	28	3111	111.11	90.62		

*: $p < .05$

Table 2 shows the three groups students’ interactions with representations in MR Geo in the test. In answering six proof questions, HG students interacted with representations for an average of 110.32 times, MG students 119.88 times, and LG students 111.11 times. These numbers showed that MG students interacted with representations most frequently, followed by LG students and HG students. ANOVA analysis demonstrated that F value was 7.79 ($p < .00075$). Scheffé post-hoc test was then used to compare the differences among groups, showing that the

interaction frequency of MG students was significantly higher than those of HG and LG students. Also, the difference between HG and LG groups was insignificant. Since MG students interacted more with multiple representations, they had more chances to build up the connections between these representations. Their learning may benefit most from these interactions. This is an interesting problem *worthy of further study*.

Students' Preferences of Representations

After performing one task each week for four weeks, students became more familiar with MR Geo and the features of each representation. In the fifth week, the authors conducted a survey on students' attitudes towards geometry learning in general and MR Geo in particular. The students were asked "*What type(s) of representation(s) do you think help your understanding of geometry proof?*" In answering, they could choose one or more of the five representations. According to the overall results in Table 3, many students preferred proof tree, formal proof, and problem description.

Table 3. Preferences on representations in different groups of students

Representation	Overall	No. of HG students	No. of MG students	No. of LG students
Problem	49 (51.0%)	8 (28.6%)	21 (52.5%)	20 (71.4%)
Static figure	42 (43.8%)	9 (32.1%)	22 (55.0%)	11 (39.3%)
Dynamic geometry figure	47 (49.0%)	10 (35.7%)	20 (50.0%)	17 (60.7%)
Formal proof	61 (63.5%)	19 (67.9%)	31 (77.5%)	11 (39.3%)
Proof tree	64 (66.7%)	18 (64.3%)	31 (77.5%)	15 (53.6%)
None of the above	4 (4.2%)	0	1 (2.5%)	3 (10.7%)

By focusing on formal proof and proof tree in Table 3, we found that HG and MG students liked the formal proof and proof tree more or less the same. This indicates that HG and MG students might know that these two representations express similar information and they can be translated into one another. However, for LG students, a graphical proof tree seems to be more revealing and a better alternative to a formal proof.

Students' relative preferences in dynamic geometry figure are quite interesting. For both HG and MG students, dynamic geometry figure is much less attractive than formal proof and proof tree. In contrast, LG students preferred dynamic geometry figure, formal proof and proof tree in this order. Our preliminary interpretation of the results from Tables 1 and 3 is as follows. HG students could understand formal proof and proof tree well enough that they did not need to consult the dynamic figure. MG and LG students did not know how to check the validity of geometric conditions with a dynamic figure. LG students liked to play with dynamic figure because they were attracted by the intriguing dynamics. Although playing with a dynamic figure did not help their understanding of a proof in general, a DGE could make geometry more interesting and attractive to them.

Students' Reactions to MR Geo

After using MR Geo, the students filled out a questionnaire to express their opinions of MR Geo. Independent *t* tests were made between the mean score of any pair of three groups for all items. Analysis indicated that there was significant difference in five items. The first item was "Did you enjoy using MR Geo?" The mean scores of MG and LG students was significantly higher than that of HG ($t = 2.74, 3.68$ for MG-HG and LG-HG, $p < .01$ and $p < .001$ respectively). The second item was "In MR Geo, DGE can be used to measure the properties of geometry figures. It can also help you understand or validate the steps of the proof." The mean score of HG students was significantly higher than those of MG and LG ($t = 3.27, 5.67$ for HG-MG and HG-LG respectively, all $p < .001$). The third item was "Overall, the MR Geo can help you read and comprehend geometry proof problems." The mean scores of HG and MG students were significantly higher than that of LG ($t = 5.42, 4.45$ for HG-LG and MG-LG respectively, all $p < .001$). The fourth item was "By observing the proof tree structure, you can identify whether proof steps can be swapped." The mean scores of HG and MG students were significantly higher than that of LG ($t = 5.25, 4.82$ for HG-LG and MG-LG respectively, all $p < .001$). The fifth item was "Through the exercises of MR Geo, geometry proofs became more interesting to you." The mean scores of MG and LG students were significantly higher than that of HG ($t = 3.27, 3.64$ for MG-HG and LG-HG respectively, all $p < .001$). The results also indicated that over 85% of the 96

ninth grade students agreed that proof tree can help them better comprehend a geometry proof. Table 4 shows the five items and their results.

Table 4. Students' responses to questionnaire

Statement	Options	Strongly agree	Agree	No comment	Disagree	Strongly disagree	Mean
1. Did you enjoy using MR Geo?		19 (19.8%)	51 (53.1%)	25 (26.1%)	1 (1.0%)	0	3.92
2. In MR Geo, DGE can be used to measure the properties of geometry figures. It can also help you understand or validate the steps of the proof.		28 (29.2%)	44 (45.8%)	22 (22.9%)	2 (2.1%)	0	4.02
3. Overall, the MR Geo can help you read and comprehend geometry proof problems.		28 (29.2%)	56 (58.3%)	11 (11.5%)	1 (1.0%)	0	4.16
4. By observing the proof tree structure, you can identify whether proof steps can be swapped.		34 (35.4%)	48 (50.0%)	12 (12.5%)	2 (2.1%)	0	4.19
5. Through the exercises of MR Geo, geometry proofs became more interesting to you.		24 (25.0%)	49 (51.1%)	22 (22.9%)	1 (1.0%)	0	3.97

In addition to the questionnaire items, we also asked students to write down their comments on MR Geo, some of which are listed as follows. "This computer program helps me understand geometry proof. In future exams, I would attempt to solve geometry proof problems" (LG). "It makes me feel that geometry proof is not that difficult" (LG). "Although I did not fully understand the geometry proofs, I believe that MR Geo is good" (LG). "I like to play with the dynamic figure, it is so funny" (LG). "I feel that geometry proof is very interesting" (MG & LG). "Compared to textbooks, this is an easier way to understand geometry proof" (MG & LG). "Previously, I didn't quite understand geometry proofs; I have now gradually developed a better understanding" (MG). "I think reading questions on paper is easier than looking at the computer display because the display confuses me" (HG). "This learning method is tiring" (HG). "DGE can help me check and verify some propositions (statements, relationships) in the formal proof or proof tree" (HG).

Conclusion

MG students interacted more frequently with representations than HG and LG students

HG students were more comfortable in using a formal proof so their interactions with other representations were not as frequent as MG students. LG students, with a weaker background in math, might not improve their performance by much even with more interactions with the representations. In this way, MG students might benefit most from their interactions with the representations in MR Geo.

Multiple representations improved students' perspective of geometry proof

Problem description, static figure, dynamic geometry figure, formal proof and proof tree had different effects on students. In particular, problem description and static figure could assist students' understanding of the problem context and only a few HG students said that dynamic geometry could help them confirm or reject a proposition. The connection between formal proof and proof tree raised students' comprehension of geometry proof. Some LG students indicated that after understanding the geometry proving process, they no longer hated geometry classes. The above results indicated that MR Geo might offer an attractive, alternative approach to geometry education with multiple representations in a computer-assisted learning environment, comparing to traditional classroom teaching.

LG and MG Students could understand a formal proof better with the help of a proof tree

Proof tree offers an outline of a complete geometry proof. Understanding the isomorphism between proof tree and formal proof can unleash the complementary and interpretation functions of both representations. By using a proof tree at the same time, students could better understand the formal proof representation. In short, students believed that using a proof tree in addition to a formal proof had a positive effect on their learning of geometry proof. This finding has some implications on the instruction of geometry proofs. A teacher in a traditional classroom can use a

proof tree to accompany a formal proof. This simple addition of a new representation might inspire LG and MG students to develop a better understanding of geometry proofs.

Encouraged by the above preliminary results, we will continue to analyze students' data on proving theorems collected in the same experiment described above. What cognitive processes did they go through during problem solving with different representations? Did they really benefit from these representations and performed better as they claimed? Another paper will try to answer these intriguing questions, which might provide invaluable insights for geometry teachers to motivate more students to develop greater interest, logical reasoning and analytic skills in geometry.

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